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Note

A Note on Riemann Sums and Improper Integrals Related to the Prime Number Theorem

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In [1], a proof of the prime number theorem is given, using the following theorem.

THEOREM A. *Let f be a real step function defined on $(0, 1]$ by $f(x) = a_n$ throughout $(1/(n+1), 1/n]$, $n = 1, 2, 3, \ldots$, namely, $f(x) = a_{[1/x]}$ throughout $(0, 1]$. If the special sequence of Riemann sums*

$$B_n = (1/n) \sum_{k=1}^{n} f(k/n), \qquad n = 1, 2, 3, \ldots,$$

converge, as $n \to \infty$, then so does the improper Riemann integral $\int_{0^-}^{1} f$, and to the same limit.

Theorem A is derived in [1] from a theorem of A. Wintner [4], which is as follows.

THEOREM B. *If the function f is Riemann integrable on $[\varepsilon, 1]$ for every ε in $(0, 1)$, and if the $\lim_{\varepsilon \to 0^-} \varepsilon \sum_{k=1}^{[1/\varepsilon]} f(k\varepsilon)$ exists, then the improper integral $\int_{0^-}^{1} f$ converges, and to the same limit.*

The proofs by A. Wintner [4] and Ingham [2], which are cited as references in [1], are difficult and deep. The purpose of this note is to present a proof of Theorem A based on a well-known result of E. Landau. It can easily be seen that the same proof is valid for Theorem B as well.

Proof of Theorem A. Let $g(x) := f(1/x)$ and

$$G(x) := \sum_{k \leqslant x} g(x/k) = \sum_{k \leqslant x} f(k/x).$$

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For $0<\varepsilon<1$,

$$\int_\varepsilon^1 f(x)\,dx=\int_\varepsilon^1 g(1/x)\,dx=\int_1^{1/\varepsilon}\frac{1}{t^2}\,g(t)\,dt$$
$$=\int_1^{1/\varepsilon}\frac{1}{t^2}\sum_{k\leqslant t}\mu(k)\,G(t/k)\,dt=\sum_{k\leqslant 1/\varepsilon}\int_k^{1/\varepsilon}\frac{\mu(k)}{t^2}\,G(t/k)\,dt,$$

where μ denotes the Möbius function. Since $G(x)=G([x])$,

$$\frac{G(x)}{x}=\frac{1}{[x]}\sum_{k\leqslant[x]}f\left(\frac{k}{[x]}\right)\cdot\frac{[x]}{x},$$

and, thus,

$$\lim_{x\to\infty}\frac{G(x)}{x}=\lim_{n\to\infty}B_n=L.$$

Hence,

$$G(x)=Lx+o(x)\qquad\text{as}\quad x\to\infty.$$

Now, using the fact that $\int_k^\infty(1/t)\,dt$ diverges, we have

$$\int_\varepsilon^1 f(x)\,dx=\sum_{k\leqslant 1/\varepsilon}\frac{\mu(k)}{k}\int_k^{1/\varepsilon}\left(\frac{L}{t}+o\left(\frac{1}{t}\right)\right)dt=L\cdot S(1/\varepsilon)+o(S(1/\varepsilon)),$$

where

$$S(1/\varepsilon)=\sum_{k\leqslant 1/\varepsilon}\frac{\mu(k)}{k}\int_k^{1/\varepsilon}\frac{1}{t}\,dt=\sum_{k\leqslant 1/\varepsilon}\frac{\mu(k)}{k}\log\frac{1}{\varepsilon}-\sum_{k\leqslant 1/\varepsilon}\frac{\mu(k)}{k}\log k$$
$$=\log\frac{1}{\varepsilon}\cdot o\left(\frac{1}{\log(1/\varepsilon)}\right)-\sum_{k\leqslant 1/\varepsilon}\frac{\mu(k)}{k}\log k,$$

by a result of Landau [3, p. 568–569]. Therefore as $\varepsilon\to 0^+$,

$$\int_\varepsilon^1 f(x)\,dx=L(o(1)-(-1))+o(1),$$

and

$$\int_{0^-}^1 f(x)\,dx=L.$$

References

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